

ON SOME q -VERSIONS OF RAMANUJAN MASTER THEOREM

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ABSTRACT. In this paper, we state some q -analogues of the famous Ramanujan's Master Theorem. As applications, some values of Jackson's q -integrals involving q -special functions are computed.

1. INTRODUCTION

In his First Quarterly Report to the Board of studies of the university of Madras in 1913, S. Ramanujan introduced a technique, known as Ramanujan Master Theorem (RMT), which provides an analytic expression for the Mellin transform of a function. The theorem is asserted as follows: if a function f can be expanded around $x = 0$ in a power series in the form

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) x^n$$

then

$$(1) \quad \int_0^{\infty} x^{s-1} f(x) dx = \frac{\pi}{\sin(\pi s)} \varphi(-s).$$

By replacing $\varphi(s)$ by $\frac{\varphi(s)}{\Gamma(s+1)}$, we obtain the following equivalent version

$$(2) \quad \int_0^{\infty} x^{s-1} \left(\sum_{n=0}^{\infty} (-1)^n \frac{\varphi(n)}{n!} x^n \right) dx = \Gamma(s) \varphi(-s).$$

This theorem was used by Ramanujan himself as a tool in computing definite integrals and infinite series. In fact, most of the examples given by Ramanujan by applying his master theorem turn out to be correct, but this later can not hold without additional assumptions, as one can easily see from the example $\varphi(s) = \sin(\pi s)$. In [3], Hardy stated a rigorous reformulation of the Ramanujan's Master Theorem. Using the residue theorem, Hardy proved the following alternative formulation:

Theorem 1. *Let f be a function having around $x = 0$ the expansion,*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) x^n,$$

with $\varphi(z)$ is an analytic function defined on a half-plane :

$$H(\delta) = \{z \in \mathbb{C} : \Re(z) \geq -\delta\}$$

for some $0 < \delta < 1$. Suppose that, for some $A < \pi$ and $P, C \in \mathbb{R}$, φ satisfies the growth condition

$$|\varphi(v + iw)| < C e^{Pv + A|w|}, \quad v + iw \in H(\delta).$$

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Then for all $0 < \text{Res} < \delta$, we have

$$(3) \quad \int_0^\infty x^{s-1} f(x) dx = \frac{\pi}{\sin(\pi x)} \varphi(-s).$$

Moreover, using the Mellin's inversion formula, Hardy proved that under the conditions of the theorem, the function f can be extended to $]0, \infty[$ and the formula (3) holds for the extension of f on $]0, \infty[$. The condition $\delta < 1$, ensures convergence of the integral in (3) and analytic continuation may be employed to valid (3) to a large strip in which the integral converges.

In view of the applications of the Ramnujan Master Theorem in computing definite integrals and infinite series, and in the interpolation field, Ismail and Stanton in [4] asked, in an open question, about the existence of a q -analogue of this famous theorem. Since the authors introduced and studied in [1] a q -analogue of the Mellin transform, this question arouses their attention and the purpose of the present paper is to give an affirmative answer. We give two q -versions of the Ramnujan Master Theorem. The proof of the first q -version is based on a Mera's theorem [8], which gives a link between the Mellin transform and its q -analogue. For the proof of the second q -version, we use the residue theorem by taking advantage of the asymptotic behavior of the q -Gamma function. Each of the two q -versions is followed by some consequences and examples.

In this paper, we adopt the following shot: Section 2 is devoted to present the elements of the quantum calculus making this paper independent. In Section 3, we recall some properties of the q -Mellin transform and we give new ones. Sections 4 and 5 are the main sections of the paper. They are devoted to state the two q -versions of the RMT and to give some of their derivative results. Finally, in Section 6, we apply the results to compute some usually q -integrals.

2. ELEMENTS OF QUANTUM CALCULUS

Throughout this text the fixed parameter of deformation is the real q selected such that $0 < q < 1$. To make this paper self containing we recall the useful q -notions. We refer to the books [2] and [5], for the notations, definitions and properties of the q -shifted factorials, the basic hypergeometric functions and the Jackson's q -integrals. For instance,

- For $a \in \mathbb{C}$, the q -shift factorials are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots, \infty.$$

- $[x]_q = \frac{1 - q^x}{1 - q}$, $x \in \mathbb{C}$, $[n]_q! = \frac{(q; q)_n}{(1 - q)^n}$, $n \in \mathbb{N}$.

- The q -hypergeometric series is defined by

$${}_r\varphi_s(a_1 \dots a_r; b_1 \dots b_s; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_k}{(b_1, b_2, \dots, b_s; q)_k (q; q)_k} \left((-1)^k q^{\frac{k(k-1)}{2}} \right)^{1+s-r} z^k,$$

where $r, s \in \mathbb{N}$, $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C}$ such that $b_1, \dots, b_s \neq 1, q^{-1}, q^{-2}, \dots$, and $(a_1, a_2, \dots, a_r; q)_k = (a_1; q)_k (a_2; q)_k \dots (a_r; q)_k$.

- The Jackson's q -integrals from 0 to a and from 0 to ∞ are given by

$$\int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \quad \int_0^{\infty} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n.$$

- The improper q -integral is defined in the following way

$$\int_0^{\frac{\infty}{A}} f(x) d_q x = (1 - q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}.$$

- Two q -analogues of the exponential function are given by (see [7])

$$(4) \quad E_q^z = \sum_{n=0}^{\infty} q^{n(n-1)} \frac{z^n}{[n]_q!} = (-(1-q)z; q)_{\infty}$$

and

$$(5) \quad e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1-q)z; q)_{\infty}}, \quad |z| < \frac{1}{1-q}.$$

- The q -gamma function is defined by

$$(6) \quad \Gamma_q(s) = \frac{(q; q)_{\infty}}{(q^s; q)_{\infty}} (1-q)^{1-s}, \quad s \neq 0, -1, -2, \dots,$$

and has the following q -integral representations:

$$(7) \quad \Gamma_q(s) = \int_0^{\frac{\infty}{1-q}} t^{s-1} E_q^{-qt} d_q t = K_q(s) \int_0^{\frac{\infty}{1-q}} t^{s-1} e_q^{-t} d_q t,$$

where

$$(8) \quad K_q(s) = \frac{(-q, -1, q)_{\infty}}{(-q^s, -q^{1-s}; q)_{\infty}}.$$

- The third Jackson's q -Bessel function, called also in some literature Heine Exton Bessel function, is defined by

$$J_{\alpha}(z; q^2) = \frac{z^{\alpha}}{(1-q^2)^{\alpha} \Gamma_{q^2}(\alpha+1)} {}_1\varphi_1(0; q^{2\alpha+2}; q^2, q^2 z^2).$$

In the end of this section, we enunciate that we will use the following sets:

- $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$.
- For $a, b \in \mathbb{R}$, ($a < b$), we write $\langle a; b \rangle = \{s \in \mathbb{C} : a < \Re(s) < b\}$.

3. q -MELLIN TRANSFORM

Hjalmar Mellin introduced the Mellin transform for a suitable function f defined over $]0, \infty[$ by

$$M(f)(s) = \int_0^{\infty} f(x) x^{s-1} dx.$$

This integral is defined in some open strip $\langle \alpha_f; \beta_f \rangle$ called fundamental band depending on the behavior of f at 0 and ∞ .

In [1], Fitouhi et al. introduced and studied a q -analogue of the Mellin transform, called q -Mellin transform, by the use of the Jackson's q -integral: for a function f defined on $\mathbb{R}_{q,+}$,

$$(9) \quad M_q(f)(s) = M_q[f(t)](s) = \int_0^{\infty} t^{s-1} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^{ns}.$$

The function $M_q(f)$ tends to $M(f)$ when q tends to 1^- , furthermore it is analytic on a band $\langle \alpha_{q,f}; \beta_{q,f} \rangle$, called the fundamental strip of $M_q(f)$, and it has the important property namely $\frac{2i\pi}{\text{Log}(q)}$ -periodic.

Other important properties and applications are proved rigorously in Fitouhi et al [1], among which we just cite

$$(10) \quad M_q[f\left(\frac{1}{t}\right)](s) = M_q(f)(-s),$$

$$(11) \quad M_q[f(x^{\rho})](s) = \left[\frac{1}{\rho}\right]_{q^{\rho}} M_{q^{\rho}}(f)\left(\frac{s}{\rho}\right), \quad \rho > 0$$

and

$$(12) \quad M_q[t^x f(t)](s) = M_q(f)(s+x).$$

In [1], the q -Mellin inversion formula is established and proved. We shall give, here, another proof based on the periodicity of this q -Mellin transform and on the Fourier series.

Theorem 2. *Let f be a function defined on $\mathbb{R}_{q,+}$ and $M_q(f)$ be its q -Mellin transform with $\langle \alpha_{q,f}; \beta_{q,f} \rangle$ as fundamental band. Then, for $c \in \langle \alpha_{q,f}; \beta_{q,f} \rangle \cap \mathbb{R}$, we have*

$$(13) \quad \forall x \in \mathbb{R}_{q,+}, \quad f(x) = \frac{\text{Log}(q)}{2i\pi(1-q)} \int_{c-\frac{i\pi}{\text{Log}(q)}}^{c+\frac{i\pi}{\text{Log}(q)}} M_q(f)(s) x^{-s} ds.$$

Proof.

Let c be a real in $\langle \alpha_{q,f}; \beta_{q,f} \rangle$. Put

$$g(t) = M_q(f)(c+it) = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^{nc} e^{int \text{Log} q}.$$

Since $M_q(f)$ is analytic on the band $\langle \alpha_{q,f}; \beta_{q,f} \rangle$ and $\frac{2i\pi}{\text{Log} q}$ -periodic, then g is continuous and $\frac{2\pi}{\text{Log}(q)}$ -periodic, on \mathbb{R} . Hence, g can be expanded in Fourier series and the convergence of the series is uniform. That is

$$g(t) = \sum_{n=-\infty}^{\infty} C_n(g) e^{int \text{Log} q},$$

where the coefficients $C_n(g)$ are given by

$$C_n(g) = \frac{\text{Log} q}{2\pi} \int_{-\frac{\pi}{\text{Log} q}}^{\frac{\pi}{\text{Log} q}} M_q(f)(c+it) e^{-int \text{Log} q} dt.$$

From the uniqueness of the development of the Fourier series, we get for all $n \in \mathbb{Z}$,

$$\begin{aligned} C_n(g) = (1-q) f(q^n) q^{nc} &= \frac{\text{Log} q}{2\pi} \int_{-\frac{\pi}{\text{Log} q}}^{\frac{\pi}{\text{Log} q}} M_q(f)(c+it) e^{-int \text{Log} q} dt \\ &= \frac{\text{Log} q}{2\pi} \int_{-\frac{\pi}{\text{Log} q}}^{\frac{\pi}{\text{Log} q}} M_q(f)(c+it) q^{-int} dt \\ &= \frac{\text{Log} q}{2i\pi} \int_{c-\frac{i\pi}{\text{Log} q}}^{c+\frac{i\pi}{\text{Log} q}} M_q(f)(s) q^{n(c-s)} ds. \end{aligned}$$

Thus, for all $x = q^n \in \mathbb{R}_{q,+}$, we have

$$f(x) = \frac{\text{Log} q}{2i\pi(1-q)} \int_{c-\frac{i\pi}{\text{Log} q}}^{c+\frac{i\pi}{\text{Log} q}} M_q(f)(s) x^{-s} ds.$$

■

The following result characterizes the functions F which are the q -Mellin transforms of functions defined on $\mathbb{R}_{q,+}$.

Theorem 3. *Let a and b be two real numbers such that $a < b$ and F be an analytic and $\frac{2i\pi}{\text{Log} q}$ -periodic function on the band $\langle a; b \rangle$. Then, F is the q -Mellin transform of the function f defined on $\mathbb{R}_{q,+}$ by*

$$f(x) = \frac{\text{Log} q}{2i\pi(1-q)} \int_{c-\frac{i\pi}{\text{Log} q}}^{c+\frac{i\pi}{\text{Log} q}} F(s) x^{-s} ds,$$

where c is an arbitrary real in $\langle a; b \rangle$.

Proof. For $a < c < b$, we consider the function h defined on \mathbb{R} by

$$h(t) = F(c + it).$$

Evidently, the function h is continuous, differentiable and $\frac{2\pi}{\text{Log}q}$ -periodic on \mathbb{R} , so it can be expanded in Fourier series as

$$h(t) = \sum_{n=-\infty}^{\infty} C_n(h) e^{int \text{Log}q},$$

where

$$\begin{aligned} C_n(h) &= \frac{\text{Log}q}{2\pi} \int_{-\frac{\pi}{\text{Log}q}}^{\frac{\pi}{\text{Log}q}} h(t) e^{-int \text{Log}q} dt = \frac{\text{Log}q}{2\pi} \int_{-\frac{\pi}{\text{Log}q}}^{\frac{\pi}{\text{Log}q}} F(c + it) q^{-int} dt \\ &= \frac{\text{Log}q}{2i\pi} \int_{c - \frac{i\pi}{\text{Log}q}}^{c + \frac{i\pi}{\text{Log}q}} F(s) q^{-n(s-c)} ds = q^{nc} \frac{\text{Log}q}{2i\pi} \int_{c - \frac{i\pi}{\text{Log}q}}^{c + \frac{i\pi}{\text{Log}q}} F(s) q^{-ns} ds \\ &= q^{nc} (1 - q) f(q^n), \end{aligned}$$

where the function f is defined on $\mathbb{R}_{q,+}$ by

$$f(q^n) = \frac{\text{Log}q}{2i\pi(1-q)} \int_{c - \frac{i\pi}{\text{Log}q}}^{c + \frac{i\pi}{\text{Log}q}} F(s) q^{-ns} ds, n \in \mathbb{Z}.$$

Then, for $s = c + it$, we have

$$\begin{aligned} F(s) &= F(c + it) = h(t) = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^{nc} e^{int \text{Log}q} \\ &= (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^{n(c+it)} = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^{n(s)} \\ &= \int_0^{\infty} x^{s-1} f(x) d_q x = M_q(f)(s). \end{aligned}$$

Finally, we remark that since F is $\frac{2i\pi}{\text{Log}q}$ -periodic and analytic on $\langle a; b \rangle$, then thanks to the Cauchy's theorem, the function f does not depend on the choice of the real c in $\langle a; b \rangle$. \blacksquare

4. FIRST q -VERSION OF RAMANUJAN MASTER THEOREM

To establish the main result of this section, we need the following result, which is recently stated by M. Mera theorem in [8] and gives a relationship between the classical Mellin and the q -Mellin transforms.

Theorem 4. *Let f a function defined on $(0, +\infty)$ and $M(f)$ be its Mellin transform with $\langle i_0, j_0 \rangle$ as fundamental strip. For $s \in \langle i_0, j_0 \rangle$ and $N \in \mathbb{N} \cup \{0\}$, we put*

$$(14) \quad M_q^N(f)(s) = -\frac{1-q}{\text{Log}q} \sum_{|n| \leq N} M(f) \left(s + i \frac{2\pi n}{\text{Log}q} \right).$$

Suppose that for a fixed σ , such that $-i_0 < \sigma < -j_0$, the sum $M_q^N(f)(\sigma + it)$ converges absolutely and uniformly for t in any compact subset of \mathbb{R} as $N \rightarrow \infty$. Then

$$\lim_{N \rightarrow \infty} M_q^N(f)(\sigma + it) = M_q(f)(\sigma + it); t \in \mathbb{R}.$$

Example

Consider the function f defined on $(0, +\infty)$ by

$$f(x) = \frac{1}{1+x}.$$

We have

$$M(f)(s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad 0 < \Re(s) < 1$$

and for all $N \in \mathbb{N} \cup \{0\}$,

$$(15) \quad M_q^N(f)(s) = -\frac{1-q}{\text{Log}q} \sum_{|n| \leq N} \frac{\pi}{\sin\left(\pi\left(s + \frac{2i\pi n}{\text{Log}q}\right)\right)}.$$

Using the relation $\Gamma(s) = \frac{\Gamma(s+2)}{s(s+1)}$ one can deduce that for $s = \sigma + it \in \langle 0; 1 \rangle$, we have

$$\left| \frac{\pi}{\sin\left(\pi\left(s + \frac{2i\pi n}{\text{Log}q}\right)\right)} \right| \leq \frac{\Gamma(\sigma+2)\Gamma(1-\sigma)}{\sigma^2 + \left(t + \frac{2\pi n}{\text{Log}q}\right)^2}.$$

So, the sum $M_q^N(f)(\sigma + it)$ converges absolutely and uniformly for t in any compact subset of \mathbb{R} . Then, from the previous theorem, we have for $0 < \Re(s) < 1$,

$$\lim_{N \rightarrow \infty} M_q^N \left[\frac{1}{1+x} \right] (s) = M_q \left[\frac{1}{1+x} \right] (s) = \frac{\Gamma_q(s)\Gamma_q(1-s)}{K_q(s)}.$$

The following theorem is the main result of this section. It is the object of the first q -version of the Ramanujan master theorem. The proof is based on the previous Mera's theorem.

Theorem 5. *Let φ be a function satisfying the conditions of Theorem 1 and $\frac{2i\pi}{\text{Log}q}$ -periodic. Then for $0 < \Re(s) < \delta$, we have*

$$(16) \quad \int_0^\infty x^{s-1} \left(\sum_{k=0}^\infty \varphi(k)(-x)^k \right) d_q x = \frac{\Gamma_q(s)\Gamma_q(1-s)}{K_q(s)} \varphi(-s),$$

where $K_q(\cdot)$ is defined by (8).

Proof.

Put

$$f(x) = \sum_{k=0}^\infty (-1)^k \varphi(k) x^k.$$

Using the Ramanujan master theorem, we get

$$M(f)(s) = \varphi(-s) \frac{\pi}{\sin(\pi s)}, \quad 0 < \Re(s) < \delta.$$

Then, thanks to the Mera's result, the periodicity of φ and the relation (15), we have for all $N \in \mathbb{N}$,

$$\begin{aligned} M_q^N(f)(s) &= -\frac{1-q}{\text{Log}q} \sum_{|n| \leq N} M(f) \left(s + \frac{2i\pi n}{\text{Log}q} \right) \\ &= -\frac{1-q}{\text{Log}q} \sum_{|n| \leq N} \frac{\pi}{\sin\left(\pi\left(s + \frac{2i\pi n}{\text{Log}q}\right)\right)} \varphi(-s) \\ &= \varphi(-s) M_q^N \left[\frac{1}{1+x} \right] (s). \end{aligned}$$

Since for $0 < \sigma < 1$, the sum $M_q^N \left[\frac{1}{1+x} \right] (\sigma + it)$ converges absolutely and uniformly for t in any compact of \mathbb{R} , then for $0 < \sigma < \delta$, the sum $M_q^N(f)(\sigma + it)$ converges absolutely and uniformly for t in any compact of \mathbb{R} and for $s = \sigma + it$, we have

$$\begin{aligned} M_q(f)(s) &= \lim_{N \rightarrow \infty} M_q^N(f)(s) = \varphi(-s) \lim_{N \rightarrow \infty} M_q^N \left[\frac{1}{1+x} \right] (s) \\ &= \varphi(-s) M_q \left[\frac{1}{1+x} \right] (s) = \frac{\Gamma_q(s)\Gamma_q(1-s)}{K_q(s)} \varphi(-s). \end{aligned}$$

■

Remark

Analytic continuation may be employed to valid (16) to a large strip in which the q -integral converges. So, in the remainder, we will not precise the strip of convergence.

Example

The function $f : x \mapsto \frac{1}{(-x; q)_\infty}$ is defined on $\Re(x) \geq 0$ and for $|x| < 1$, we have

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) x^n,$$

where $\varphi(s) = \frac{(q^{s+1}; q)_\infty}{(q; q)_\infty}$.

It is easy to see that the function φ is entire, $\frac{2i\pi}{\text{Log} q}$ -periodic on \mathbb{C} and for $\Re(s) > -\delta > -1$, we have

$$|\varphi(s)| \leq \frac{(-1; q)_\infty}{(q; q)_\infty}.$$

So, for $0 < \Re(s) < \delta$, we have

$$M_q(f)(s) = \frac{(q^{1-s}; q)_\infty}{(q; q)_\infty} \frac{\Gamma_q(s) \Gamma_q(1-s)}{K_q(s)} = (1-q)^s \frac{\Gamma_q(s)}{K_q(s)}.$$

Since the q -integral $M_q(f)$ converges for $\Re(s) > 0$, then this relation can be extended to $\Re(s) > 0$. As direct consequence, we get

$$\int_0^{\frac{\infty}{1-q}} \frac{x^{s-1}}{(-(1-q)x; q)_\infty} d_q x = \frac{\Gamma_q(s)}{K_q(s)}.$$

Proposition 1. *Let φ be a function satisfying the conditions of Theorem 5, and put*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^n}{(q; q)_n}.$$

Then

$$(17) \quad M_q(f)(s) = (1-q)^s \varphi(-s) \frac{\Gamma_q(s)}{K_q(s)}.$$

Proof.

Remark that

$$\forall n \in \mathbb{N}, \quad \frac{\varphi(n)}{(q; q)_n} = \varphi(n) \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty}$$

and put

$$\psi(s) = \varphi(s) \frac{(q^{s+1}; q)_\infty}{(q; q)_\infty}.$$

If φ satisfies the conditions of Theorem 5, then the function ψ is $\frac{2i\pi}{\text{Log}(q)}$ -periodic and analytic on $H(\delta)$, and for $s = v + iw \in H(\delta)$, we have $v \geq -\delta \geq -1$ and

$$|\psi(s)| = \left| \varphi(s) \frac{(q^{s+1}; q)_\infty}{(q; q)_\infty} \right| \leq C \frac{(-1; q)_\infty}{(q; q)_\infty} e^{Pv + A|w|}.$$

Then, ψ satisfies the conditions of Theorem 5, so for $0 < \Re(s) < \delta$, we have

$$M_q(f)(s) = M_q \left[\sum_{n=0}^{\infty} (-1)^n \psi(n) x^n \right] (s) = \frac{\Gamma_q(s) \Gamma_q(1-s)}{K_q(s)} \psi(-s) = (1-q)^s \varphi(-s) \frac{\Gamma_q(s)}{K_q(s)}.$$

■

Corollary 1. *Let φ be an analytic function on $H(\delta)$ such that the function $s \mapsto (1 - q)^s \varphi(s)$ satisfies the conditions of Theorem 5. Then, for $0 < \Re(s) < \delta$, we have*

$$(18) \quad M_q \left[\sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^n}{[n]_q!} \right] = \varphi(-s) \frac{\Gamma_q(s)}{K_q(s)}.$$

In the particular case, $\frac{\text{Log}(1-q)}{\text{Log}q} \in \mathbb{Z}$, we have:

Corollary 2. *If φ is a function satisfying the conditions of Theorem 5, then for $0 < \Re(s) < \delta$, we have*

$$(19) \quad M_q \left[\sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^n}{[n]_q!} \right] = \varphi(-s) \frac{\Gamma_q(s)}{K_q(s)}.$$

In particular we have

$$M_q \left[\frac{1}{(-(1-q)x; q)_{\infty}} \right] (s) = \frac{\Gamma_q(s)}{K_q(s)}.$$

Proposition 2. *Let $\alpha > 0$, $p \in \mathbb{N} \cup \{0\}$ and φ be an analytic function on $H(\delta)$ such that the function $s \mapsto (1 - q^\alpha)^s \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic and satisfies the conditions of Theorem 1. Put*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^{\alpha n}}{[n+p]_{q^\alpha}!}.$$

Then,

$$(20) \quad M_q(f)(s) = \left[\frac{1}{\alpha} \right]_{q^\alpha} \varphi \left(-\frac{s}{\alpha} \right) \frac{\Gamma_{q^\alpha} \left(1 - \frac{s}{\alpha} \right)}{\Gamma_{q^\alpha} \left(1 - \frac{s}{\alpha} + p \right)} \frac{\Gamma_{q^\alpha} \left(\frac{s}{\alpha} \right)}{K_{q^\alpha} \left(\frac{s}{\alpha} \right)}$$

Proof.

On the one hand, we have

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^{\alpha n}}{[n+p]_{q^\alpha}!} = h(x^\alpha),$$

where

$$h(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^n}{[n+p]_{q^\alpha}!}.$$

So, using the properties of the q -Mellin transform, we get

$$(21) \quad M_q(f)(s) = \left[\frac{1}{\alpha} \right]_{q^\alpha} M_{q^\alpha}(h) \left(\frac{s}{\alpha} \right).$$

On the other hand, since the function $s \mapsto \frac{\varphi(s)}{\Gamma_{q^\alpha}(1+s+p)} = \frac{(q^{\alpha(s+p+1)}; q^\alpha)_{\infty}}{(q^\alpha; q^\alpha)_{\infty}} (1 - q^\alpha)^{s+p} \varphi(s)$ satisfies the conditions of Theorem 1 and it is $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic, then by applying Theorem 5, we obtain

$$M_{q^\alpha}(h)(s) = \frac{\varphi(-s)}{\Gamma_{q^\alpha}(1-s+p)} \frac{\Gamma_{q^\alpha}(1-s) \Gamma_{q^\alpha}(s)}{K_{q^\alpha}(s)}.$$

Hence, the result follows from the relation (21). ■

For the particular case $p = 0$, we obtain the following result:

Corollary 3. *Let $\alpha > 0$ and φ be a function satisfying the condition of Theorem 1 such that the function $s \mapsto (1 - q^\alpha)^s \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic. Then,*

$$(22) \quad M_q \left[\sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^{\alpha n}}{[n]_{q^\alpha}!} \right] (s) = \left[\frac{1}{\alpha} \right]_{q^\alpha} \varphi \left(-\frac{s}{\alpha} \right) \frac{\Gamma_{q^\alpha} \left(\frac{s}{\alpha} \right)}{K_{q^\alpha} \left(\frac{s}{\alpha} \right)}.$$

Proposition 3. *Let $\alpha \in \mathbb{N}$, $p \in \mathbb{N} \cup \{0\}$ and φ be a function satisfying the condition of Theorem 1 and such that the function $s \mapsto (1-q)^{\alpha s} \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic. Then, the q -Mellin transform of the function*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^{\alpha n}}{[\alpha n + p]_q!}$$

is given by

$$(23) \quad M_q(f)(s) = \left[\frac{1}{\alpha} \right]_{q^\alpha} \varphi \left(-\frac{s}{\alpha} \right) \frac{\Gamma_{q^\alpha} \left(1 - \frac{s}{\alpha} \right)}{\Gamma_q(1-s+p)} \frac{\Gamma_{q^\alpha} \left(\frac{s}{\alpha} \right)}{K_{q^\alpha} \left(\frac{s}{\alpha} \right)}.$$

Proof.

Following the same steps of the proof of Proposition 2 by replacing the function $s \mapsto \frac{\varphi(s)}{\Gamma_q(1+s+p)}$ by the functions $s \mapsto \frac{\varphi(s)}{\Gamma_q(1+\alpha s+p)}$, we get the desired result. \blacksquare

5. SECOND q -VERSION OF THE RAMANUJAN MASTER THEOREM

In this section, we use the residue theorem, to prove a new q -version of the Ramanujan master Theorem.

Theorem 6. *Let φ be an analytic function defined on a half-plane*

$$H(\delta) = \{z \in \mathbb{C} : \Re(z) \geq -\delta\}$$

for some $0 < \delta < 1$. Suppose that φ is $\frac{2i\pi}{\text{Log}(q)}$ -periodic, and there exist a continuous function φ_1 on \mathbb{R} and real $r \in \mathbb{R}$ such that

$$(24) \quad \forall z = u + iw \in H(\delta), \quad |\varphi(u + iw)| \leq e^{r|u|} \varphi_1(w).$$

Then, for all $s \in \langle 0, \delta \rangle$, we have

$$(25) \quad \int_0^\infty x^{s-1} \left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \varphi(n) x^n \right) d_q x = \Gamma_q(s) \Gamma_q(1-s) \varphi(-s).$$

Proof.

Let φ be a function satisfying the conditions of the theorem 1 and $x = q^m \in \mathbb{R}_{q,+}$. Then the function ψ defined by

$$\psi(s) = \Gamma_q(s) \Gamma_q(1-s) \varphi(-s) x^{-s}$$

is $\frac{2i\pi}{\text{Log}(q)}$ -periodic and analytic on $\langle 0, \delta \rangle$ and meromorphic on the half-plane $\Re(s) \leq \delta$, with simple poles at $0, -1, -2, -3, \dots$. Furthermore, for all $n \in \mathbb{N} \cap \{0\}$, we have

$$\begin{aligned} \text{Res}(\psi, -n) &= \lim_{s \rightarrow -n} (s+n) \psi(s) \\ &= \lim_{s \rightarrow -n} [(s+n) \Gamma_q(s) \Gamma_q(1-s) \varphi(-s) x^{-s}] \\ &= \frac{1-q}{\text{Log}(q)} q^{\frac{n(n+1)}{2}} (-1)^{n+1} \varphi(n) x^n. \end{aligned}$$

The conditions on the function φ and the presence of the factor $q^{\frac{n(n+1)}{2}}$ show that the series

$$\sum_{n \geq 0} \frac{1-q}{\text{Log}(q)} q^{\frac{n(n+1)}{2}} (-1)^{n+1} \varphi(n) x^n \text{ converges.}$$

On the other hand, for $N \in \mathbb{N}$ and $0 < \sigma < \delta$, the application of the residue theorem for the

function ψ using the rectangular contour C_N defined by the segments $\left[\sigma - \frac{i\pi}{\text{Log}(q)}, \sigma + \frac{i\pi}{\text{Log}(q)} \right]$, $\left[\sigma + \frac{i\pi}{\text{Log}(q)}, -N + \frac{1}{2} + \frac{i\pi}{\text{Log}(q)} \right]$, $\left[-N - \frac{1}{2} + \frac{i\pi}{\text{Log}(q)}, -N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)} \right]$, $\left[-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}, \sigma - \frac{i\pi}{\text{Log}(q)} \right]$,

and $\left[-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}, \sigma - \frac{i\pi}{\text{Log}(q)}\right]$ yields

$$(26) \quad -2i\pi \sum_{n=0}^N \text{Res}(\psi, -n) = \int_{C_N} \psi(s) ds,$$

where

$$\begin{aligned} \int_{C_N} \psi(s) ds &= \int_{\sigma - \frac{i\pi}{\text{Log}(q)}}^{\sigma + \frac{i\pi}{\text{Log}(q)}} \psi(s) ds + \int_{\sigma + \frac{i\pi}{\text{Log}(q)}}^{-N - \frac{1}{2} + \frac{i\pi}{\text{Log}(q)}} \psi(s) ds + \int_{-N - \frac{1}{2} + \frac{i\pi}{\text{Log}(q)}}^{-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}} \psi(s) ds \\ &\quad + \int_{-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}}^{\sigma - \frac{i\pi}{\text{Log}(q)}} \psi(s) ds. \end{aligned}$$

The periodicity and the analyticity of the function ψ imply that the sum of the second and the fourth integrals is zero. Moreover, the third integral verifies

$$\begin{aligned} \int_{-N - \frac{1}{2} + \frac{i\pi}{\text{Log}(q)}}^{-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}} \psi(s) ds &= i \int_{\frac{\pi}{\text{Log}(q)}}^{-\frac{\pi}{\text{Log}(q)}} \psi(-N - \frac{1}{2} + it) dt \\ &= i \int_{\frac{\pi}{\text{Log}(q)}}^{-\frac{\pi}{\text{Log}(q)}} \Gamma_q(-N - \frac{1}{2} + it) \Gamma_q(N + \frac{3}{2} - it) \varphi(N + \frac{1}{2} - it) x^{N + \frac{1}{2} - it} dt \end{aligned}$$

and

$$\left| \int_{-N - \frac{1}{2} + \frac{i\pi}{\text{Log}(q)}}^{-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}} \psi(s) ds \right| \leq \int_{\frac{\pi}{\text{Log}(q)}}^{-\frac{\pi}{\text{Log}(q)}} \left| \Gamma_q(-N - \frac{1}{2} + it) \Gamma_q(N + \frac{3}{2} - it) \right| |\varphi(N + \frac{1}{2} - it)| x^{N + \frac{1}{2}} dt.$$

But for all $t \in \left[\frac{\pi}{\text{Log}(q)}, -\frac{\pi}{\text{Log}(q)}\right]$ and all $N \in \mathbb{N}$, we have

$$\begin{aligned} \Gamma_q(-N - \frac{1}{2} + it) \Gamma_q(N + \frac{3}{2} - it) &= \frac{(1-q)^N}{(q^{-N - \frac{1}{2} + it}; q)_N} \Gamma_q(-\frac{1}{2} + it) \frac{(q^{\frac{3}{2} - it}; q)_N}{(1-q)^N} \Gamma_q(\frac{3}{2} - it) \\ &= \frac{(q^{\frac{3}{2} - it}; q)_N}{(q^{-N - \frac{1}{2} + it}; q)_N} \Gamma_q(-\frac{1}{2} + it) \Gamma_q(\frac{3}{2} - it). \end{aligned}$$

Since

$$|(q^{\frac{3}{2} - it}; q)_N| = \prod_{k=0}^{N-1} |1 - q^{\frac{3}{2} - it + k}| \leq \prod_{k=0}^{N-1} (1 + q^{\frac{3}{2} + k}) \leq (-q^{\frac{3}{2}}; q)_\infty$$

and

$$\begin{aligned} |(q^{-N - \frac{1}{2} + it}; q)_N| &= \prod_{k=0}^{N-1} |1 - q^{-N - \frac{1}{2} + it + k}| \geq \prod_{k=0}^{N-1} (q^{-\frac{1}{2} - N + k} - 1) = \prod_{k=0}^{N-1} q^{-\frac{1}{2} - N + k} (1 - q^{\frac{1}{2} + N - k}) \\ &\geq q^{-\frac{N}{2} - \frac{N(N+1)}{2}} \frac{(q^{\frac{1}{2}}; q)_{N+1}}{1 - q^{\frac{1}{2}}} \geq q^{-\frac{N(N+2)}{2}} \frac{(q^{\frac{1}{2}}; q)_\infty}{1 - q^{\frac{1}{2}}}, \end{aligned}$$

then

$$\left| \Gamma_q(-N - \frac{1}{2} + it) \Gamma_q(N + \frac{3}{2} - it) \right| \leq q^{\frac{N(N+2)}{2}} \frac{(-q^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}}; q)_\infty} \left| \Gamma_q(\frac{1}{2} + it) \Gamma_q(\frac{3}{2} - it) \right|.$$

Hence,

$$\begin{aligned} &\left| \int_{-N - \frac{1}{2} + \frac{i\pi}{\text{Log}(q)}}^{-N - \frac{1}{2} - \frac{i\pi}{\text{Log}(q)}} \psi(s) ds \right| \\ &\leq q^{\frac{N(N+2)}{2}} \frac{(-q^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}}; q)_\infty} \int_{\frac{\pi}{\text{Log}(q)}}^{-\frac{\pi}{\text{Log}(q)}} \left| \Gamma_q(\frac{1}{2} + it) \Gamma_q(\frac{3}{2} - it) \right| |\varphi(N + \frac{1}{2} - it)| x^{N + \frac{1}{2}} dt \\ &\leq q^{\frac{N(N+2)}{2}} x^{N + \frac{1}{2}} e^{r(N + \frac{1}{2})} \frac{(-q^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}}; q)_\infty} \int_{\frac{\pi}{\text{Log}(q)}}^{-\frac{\pi}{\text{Log}(q)}} \left| \Gamma_q(\frac{1}{2} + it) \Gamma_q(\frac{3}{2} - it) \right| |\varphi_1(-t)| dt. \end{aligned}$$

Since the function $t \mapsto \left| \Gamma_q\left(\frac{1}{2} + it\right) \Gamma_q\left(\frac{3}{2} - it\right) \right| \varphi_1(-t)$ is continuous on $\left[\frac{\pi}{\text{Log}(q)}, -\frac{\pi}{\text{Log}(q)} \right]$ and independent of N , and since $0 < q < 1$, we have

$$q^{\frac{N(N+2)}{2}} x^{N+\frac{1}{2}} e^{r(N+\frac{1}{2})} = q^{\frac{N(N+2)}{2}} q^{m(N+\frac{1}{2})} e^{r(N+\frac{1}{2})} \rightarrow 0 \text{ as } N \rightarrow +\infty,$$

then,

$$\lim_{N \rightarrow +\infty} \int_{-N-\frac{1}{2}-\frac{i\pi}{\text{Log}(q)}}^{-N-\frac{1}{2}+\frac{i\pi}{\text{Log}(q)}} \psi(s) ds = 0.$$

So, taking the limit as $N \rightarrow +\infty$, in (26), we obtain

$$\int_{\sigma-\frac{i\pi}{\text{Log}(q)}}^{\sigma+\frac{i\pi}{\text{Log}(q)}} \psi(s) ds = \frac{2i\pi(1-q)}{\text{Log}(q)} \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-1)^n \varphi(n) x^n.$$

Finally, for all $x \in \mathbb{R}_{q,+}$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-1)^n \varphi(n) x^n &= \frac{\text{Log}(q)}{2i\pi(1-q)} \int_{\sigma-\frac{i\pi}{\text{Log}(q)}}^{\sigma+\frac{i\pi}{\text{Log}(q)}} \psi(s) ds \\ &= \frac{\text{Log}(q)}{2i\pi(1-q)} \int_{\sigma-\frac{i\pi}{\text{Log}(q)}}^{\sigma+\frac{i\pi}{\text{Log}(q)}} \Gamma_q(s) \Gamma_q(1-s) \varphi(-s) x^{-s} ds. \end{aligned}$$

Since the function $s \mapsto \Gamma_q(s) \Gamma_q(1-s) \varphi(-s)$ is analytic on $\langle 0, \delta \rangle$ and $\frac{2i\pi}{\text{Log}(q)}$ -periodic, then from Theorem 3, for all $s \in \langle 0, \delta \rangle$, we have

$$\begin{aligned} \Gamma_q(s) \Gamma_q(1-s) \varphi(-s) &= M_q \left[\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-1)^n \varphi(n) x^n \right] (s) \\ &= \int_0^{\infty} x^{s-1} \left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-1)^n \varphi(n) x^n \right) d_q x. \end{aligned}$$

■

Example

Consider the function defined by

$$f(x) = E_q^{-qx} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \frac{x^n}{(q; q)_n} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \varphi(n) x^n,$$

with $\varphi(s) = \frac{(q^{s+1}; q)_{\infty}}{(q; q)_{\infty}}$.

The function φ is analytic and $\frac{2i\pi}{\text{Log}(q)}$ -periodic on \mathbb{C} and for $\Re(s) > -\delta > -1$, we have

$$|\varphi(s)| \leq \frac{(-1; q)_{\infty}}{(q; q)_{\infty}}.$$

Then, from the previous theorem, we have for $0 < \Re(s) < \delta$,

$$(27) \quad M_q [E_q^{-qx}] (s) = \Gamma_q(s) \Gamma_q(1-s) \varphi(-s) = (1-q)^s \Gamma_q(s).$$

In particular, if the parameter q satisfies $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in \mathbb{Z}$, then

$$M_q [E_q^{-q(1-q)x}] (s) = \Gamma_q(s).$$

A generalization of this example is given in the following result.

Proposition 4. *Let φ be an analytic function satisfying the hypothesis of Theorem 6 and f be a function such that*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \varphi(n) \frac{x^n}{(q; q)_n}.$$

Then for $0 < \Re(s) < \delta$,

$$(28) \quad M_q(f)(s) = (1-q)^s \varphi(-s) \Gamma_q(s).$$

Proof. It is easy to see that

$$f(x) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \varphi(n) \frac{x^n}{(q; q)_n} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \psi(n) x^n,$$

$$\text{with } \psi(s) = \frac{(q^{s+1}; q)_{\infty}}{(q; q)_{\infty}} \varphi(s).$$

Since φ satisfies the conditions of Theorem 6, and $s \mapsto \frac{(q^{s+1}; q)_{\infty}}{(q; q)_{\infty}}$ is entire, $\frac{2i\pi}{\text{Log}(q)}$ -periodic on \mathbb{C} and for $\Re(s) > -\delta > -1$, we have

$$\left| \frac{(q^{s+1}; q)_{\infty}}{(q; q)_{\infty}} \right| \leq \frac{(-1; q)_{\infty}}{(q; q)_{\infty}},$$

then ψ satisfies the conditions of Theorem 6, and for $0 < \Re(s) < 1$, we have

$$M_q(f)(s) = M_q \left[\sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \psi(n) x^n \right] (s) = \Gamma_q(s) \Gamma_q(1-s) \psi(-s) = (1-q)^s \varphi(-s) \Gamma_q(s).$$

■

Corollary 4. *Let φ be an analytic function on $H(\delta)$ such that the function $s \mapsto (1-q)^s \varphi(s)$ satisfies the hypothesis of Theorem 6 and consider*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \varphi(n) \frac{x^n}{[n]_q!}.$$

Then, for $0 < \Re(s) < \delta$,

$$(29) \quad M_q(f)(s) = \varphi(-s) \Gamma_q(s).$$

Remark. In the case that $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in \mathbb{Z}$, we have $1-q \in \mathbb{R}_{q,+}$. So, if a function φ satisfies the conditions of Theorem 6, then $s \mapsto (1-q)^s \varphi(s)$ satisfies them and from the previous corollary, for $0 < \Re(s) < \delta$, we have

$$(30) \quad M_q \left[\sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} \varphi(n) \frac{x^n}{[n]_q!} \right] (s) = \varphi(-s) \Gamma_q(s).$$

Proposition 5. *Let $\alpha > 0$, $p \in \mathbb{N} \cup \{0\}$ and φ be an analytic function on $H(\delta)$ satisfying the condition (24) such that the function $s \mapsto (1-q^\alpha)^s \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic and put*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^{\alpha n}}{[n+p]_{q^\alpha}!}.$$

Then,

$$(31) \quad M_q(f)(s) = \left[\frac{1}{\alpha} \right]_{q^\alpha} \varphi \left(-\frac{s}{\alpha} \right) \frac{\Gamma_{q^\alpha} \left(1 - \frac{s}{\alpha} \right)}{\Gamma_{q^\alpha} \left(1 - \frac{s}{\alpha} + p \right)} \Gamma_{q^\alpha} \left(\frac{s}{\alpha} \right).$$

Proof.

We have

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^{\alpha n}}{[n+p]_{q^\alpha}!} = h(x^\alpha),$$

with

$$h(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^n}{[n+p]_{q^\alpha}!}.$$

But, under the hypotheses of the proposition, the function $s \mapsto \frac{\varphi(s)}{\Gamma_{q^\alpha}(1+s+p)}$ is analytic, $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic on $H(\delta)$ and satisfies the condition (24). Then, from Theorem 6, we have for $0 < \Re(s) < \delta$,

$$M_{q^\alpha}(h)(s) = \varphi(-s) \frac{\Gamma_{q^\alpha}(1-s)}{\Gamma_{q^\alpha}(1-s+p)} \Gamma_{q^\alpha}(s).$$

Finally, the relation (21) achieves the proof. ■

For the particular case $p = 0$, if φ satisfies the conditions of the proposition, we get

$$(32) \quad M_q \left[\sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^{\alpha n}}{[n]_{q^\alpha}!} \right] = \left[\frac{1}{\alpha} \right]_{q^\alpha} \varphi \left(-\frac{s}{\alpha} \right) \Gamma_{q^\alpha} \left(\frac{s}{\alpha} \right).$$

Proposition 6. *Let $\alpha \in \mathbb{N}$, $p \in \mathbb{N} \cup \{0\}$ and φ be an analytic function on $H(\delta)$ satisfying the condition (24) such that the function $s \mapsto (1-q)^{\alpha s} \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic. Then, the q -Mellin transform of the function*

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^{\alpha n}}{[\alpha n + p]_q!}$$

is given by

$$(33) \quad M_q(f)(s) = \left[\frac{1}{\alpha} \right]_{q^\alpha} \varphi \left(-\frac{s}{\alpha} \right) \frac{\Gamma_{q^\alpha} \left(1 - \frac{s}{\alpha} \right)}{\Gamma_q(1-s+p)} \Gamma_{q^\alpha} \left(\frac{s}{\alpha} \right).$$

Proof.

We have

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^{\alpha n}}{[\alpha n + p]_q!} = h(x^\alpha),$$

where

$$h(x) = \sum_{n=0}^{\infty} (-1)^n \varphi(n) q^{\alpha \frac{n(n+1)}{2}} \frac{x^n}{[\alpha n + p]_q!}.$$

Under the hypotheses of the proposition, the function $s \mapsto \frac{\varphi(s)}{\Gamma_q(1+\alpha s+p)}$ is analytic, $\frac{2i\pi}{\text{Log}(q^\alpha)}$ -periodic on $H(\delta)$ and satisfies the condition (24). Then, from Theorem 6, we have for $0 < \Re(s) < \delta$,

$$M_{q^\alpha}(h)(s) = \frac{\varphi(-s)}{\Gamma_q(1-\alpha s+p)} \Gamma_{q^\alpha}(1-s) \Gamma_{q^\alpha}(s).$$

Hence, the result follows from the relation (21).

6. EXAMPLES

In this section, we shall apply the two q -version of the Ramanujan master theorem to find the q -Mellin transform of some q -special functions.

Example 1:

$$\text{Let } a \in \mathbb{R}_{q,+}, \gamma > 0 \text{ and } f(x) = \frac{(-axq^\gamma; q)_\infty}{(-ax; q)_\infty}.$$

By using the q -Binomial formula, we get

$$\begin{aligned} \frac{(-axq^\gamma; q)_\infty}{(-ax; q)_\infty} &= \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n}{(q; q)_n} (-ax)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma_q(\gamma+n)}{\Gamma_q(\gamma)} a^n \frac{x^n}{[n]_q!}. \end{aligned}$$

We consider the function φ defined by

$$(34) \quad \varphi(s) = \frac{\Gamma_q(\gamma+s)}{\Gamma_q(\gamma)} a^s; \quad \Re(s) > -\gamma.$$

Let $0 < \delta < \min(1, \gamma)$. It is easy to see that the function φ is analytic on $H(\delta)$, and the function $s \mapsto (1-q)^s \varphi(s)$ is analytic and $\frac{2i\pi}{\text{Log}(q)}$ -periodic on $H(\delta)$. Furthermore, for $s = u + iv \in H(\delta)$,

we have

$$\begin{aligned} |(1-q)^s \varphi(s)| &= \left| (1-q)^s \frac{\Gamma_q(\gamma+s)}{\Gamma_q(\gamma)} a^s \right| = \frac{1-q}{\Gamma_q(\gamma)} |(q^{s+\gamma}; q)_\infty| a^u \\ &\leq \frac{1-q}{\Gamma_q(\gamma)} (-q^{\gamma-\delta}; q)_\infty e^{u \text{Log}(a)}. \end{aligned}$$

So from Corollary 1, we obtain for $0 < \Re(s) < \delta$,

$$(35) \quad M_q(f)(s) = \varphi(-s) \frac{\Gamma_q(s)}{K_q(s)} = \frac{\Gamma_q(\gamma-s)}{\Gamma_q(\gamma)} a^{-s} \frac{\Gamma_q(s)}{K_q(s)} = \frac{a^{-s}}{K_q(s)} B_q(\gamma-s, s).$$

Example 2:

Consider the q -cosine function defined by

$$\cos(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}.$$

We assume that $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in 2\mathbb{Z}$ and we consider the function $\varphi(s) = 1$.

So, the function $s \mapsto (1-q)^{2s} \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^2)}$ -periodic. Then by applying Proposition 6, for $\alpha = 2$ and $p = 0$, we obtain

$$(36) \quad M_q[\cos(x; q^2)](s) = \left[\frac{1}{2} \right]_{q^2} \frac{\Gamma_{q^2}\left(1 - \frac{s}{2}\right) \Gamma_{q^2}\left(\frac{s}{2}\right)}{\Gamma_q(1-s)}.$$

Example 3:

We assume that $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in 2\mathbb{Z}$ and we consider the function q -sine defined by

$$f(x) = \sin(x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!} = xh(x),$$

where

$$h(x) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n+1]_q!}.$$

Since, $\frac{\text{Log}(1-q)}{\text{Log}(q)} \in 2\mathbb{Z}$, then the function $s \mapsto (1-q)^{2s}$ is $\frac{2i\pi}{\text{Log}(q^2)}$ -periodic. So, by application of Proposition 6, with $\alpha = 2$, $p = 1$ and $\varphi(s) = 1$, we obtain

$$(37) \quad M_q(h)(s) = \left[\frac{1}{2} \right]_{q^2} \frac{\Gamma_{q^2}\left(1 - \frac{s}{2}\right) \Gamma_{q^2}\left(\frac{s}{2}\right)}{\Gamma_q(2-s)}.$$

Therefore, by using the properties of the q -Mellin transform, we obtain

$$(38) \quad M_q(\sin(x; q^2))(s) = M_q(h)(s+1) = \left[\frac{1}{2} \right]_{q^2} \frac{\Gamma_{q^2}\left(\frac{1-s}{2}\right) \Gamma_{q^2}\left(\frac{1+s}{2}\right)}{\Gamma_q(1-s)}.$$

Example 4:

We consider the third Jackson's q -Bessel function defined by

$$\begin{aligned} f(x) = J_\nu(x; q^2) &= \frac{x^\nu}{(1-q^2)^\nu \Gamma_{q^2}(\nu+1)} {}_1\varphi_1(0; q^{2\nu+2}; q^2; q^2 x^2) \\ &= \frac{x^\nu}{(1-q^2)^\nu} \sum_{n=0}^{\infty} (-1)^n \frac{(1-q^2)^{-2n}}{\Gamma_{q^2}(n+\nu+1)} q^{n(n+1)} \frac{x^{2n}}{[n]_{q^2}!} \\ &= \frac{x^\nu}{(1-q^2)^\nu} h(x), \end{aligned}$$

where

$$h(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(1-q^2)^{-2n}}{\Gamma_{q^2}(n+\nu+1)} q^{n(n+1)} \frac{x^{2n}}{[n]_{q^2}!}.$$

We have

$$M_q(f)(s) = \frac{1}{(1-q^2)^\nu} M_q(x^\nu h(x))(s) = \frac{1}{(1-q^2)^\nu} M_q(h(x))(s+\nu).$$

On the other hand, the function $\varphi(s) = \frac{(1-q^2)^{-2s}}{\Gamma_{q^2}(s+\nu+1)}$, is entire on \mathbb{C} and for $0 < \delta < 1$, we have for $\Re(s) > -\delta > -1$,

$$|\varphi(s)| = \left| (1-q^2)^{\nu-s} \frac{(q^{s+\nu+1}; q^2)_\infty}{(q^2; q^2)_\infty} \right| \leq (1-q^2)^\nu \frac{(-q^\nu; q^2)_\infty}{(q^2; q^2)_\infty} e^{-\text{Log}(1-q^2)\Re(s)}.$$

Furthermore, it is easy to verify that the function $s \mapsto (1-q^2)^s \varphi(s)$ is $\frac{2i\pi}{\text{Log}(q^2)}$ -periodic. So, by application of the Proposition 5 for $\alpha = 2$ and $p = 0$, we obtain

$$(39) \quad M_q(h)(s) = \left[\frac{1}{2} \right]_{q^2} \varphi\left(-\frac{s}{2}\right) \Gamma_{q^2}\left(\frac{s}{2}\right) = \left[\frac{1}{2} \right]_{q^2} \frac{(1-q^2)^s}{\Gamma_{q^2}\left(-\frac{s}{2} + \nu + 1\right)} \Gamma_{q^2}\left(\frac{s}{2}\right).$$

Therefore

$$(40) \quad M_q[J_\nu(x; q^2)](s) = \left[\frac{1}{2} \right]_{q^2} (1-q^2)^s \frac{\Gamma_{q^2}\left(\frac{s+\nu}{2}\right)}{\Gamma_{q^2}\left(\frac{-s+\nu+2}{2}\right)}.$$

Example 5:

Let

$$\begin{aligned} f(x) &= {}_r\varphi_r(q^{a_1}, q^{a_2}, \dots, q^{a_r}; q^{b_1}, q^{b_2}, \dots, q^{b_r}; q; qx) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(q^{a_1}; q)_n (q^{a_2}; q)_n \dots (q^{a_r}; q)_n}{(q^{b_1}; q)_n (q^{b_2}; q)_n \dots (q^{b_r}; q)_n} q^{\frac{n(n+1)}{2}} \frac{x^n}{(q; q)_n} \\ &= \sum_{n=0}^{\infty} (-1)^n \varphi(n) \frac{x^n}{(q; q)_n}, \end{aligned}$$

with

$$\varphi(s) = \frac{(q^{a_1}, q)_\infty (q^{a_2}, q)_\infty \dots (q^{a_r}, q)_\infty}{(q^{b_1}, q)_\infty (q^{b_2}, q)_\infty \dots (q^{b_r}, q)_\infty} \frac{(q^{b_1+s}, q)_\infty (q^{b_2+s}, q)_\infty \dots (q^{b_r+s}, q)_\infty}{(q^{a_1+s}, q)_\infty (q^{a_2+s}, q)_\infty \dots (q^{a_r+s}, q)_\infty}.$$

The function φ is analytic on the half-plane $\Re(s) > -\min_{1 \leq k \leq r} (a_k)$ and it is $\frac{2i\pi}{\text{Log}(q)}$ periodic. So,

$$\begin{aligned} M_q(f)(s) &= (1-q)^s \varphi(-s) \Gamma_q(s) \\ &= (1-q)^s \frac{\Gamma_q(a_1-s) \Gamma_q(a_2-s) \dots \Gamma_q(a_r-s)}{\Gamma_q(b_1-s) \Gamma_q(b_2-s) \dots \Gamma_q(b_r-s)} \frac{\Gamma_q(b_1) \Gamma_q(b_2) \dots \Gamma_q(b_r)}{\Gamma_q(a_1) \Gamma_q(a_2) \dots \Gamma_q(a_r)} \Gamma_q(s). \end{aligned}$$

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